Institut für Informatik
Prof. Dr. Heiko Röglin
Michael Etscheid

Probabilistic Analysis
of Algorithms
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Problem Set 8

Let $G = (V, E)$ be a complete undirected graph and for $e \in E$ let the edge length $c(e)$ be drawn independently from the other edge lengths according to a densitiy function $f_e\colon [0,1] \to [0, \phi]$. We want to find a good solution for this instance of the traveling salesman problem (TSP) using the 2-*Opt* heuristic.

For this, we start with an arbitrary tour and perform 2-*changes*: Remove two edges from the tour which do not share a common vertex and reconnect the two resulting connected components in the only other possible way provided that this change decreases the length of the tour.

If replacing $e_1, e_2 \in E$ by $e_3, e_4 \in E$ is an improving 2-change for any tour which contains the edges e_1 and e_2, the total length of the tour decreases by

$$\Delta(e_1, e_2, e_3, e_4) = c(e_1) + c(e_2) - c(e_3) - c(e_4)\,.$$

For all other $e_1, e_2, e_3, e_4 \in E$, let $\Delta(e_1, e_2, e_3, e_4) = \infty$. Define Δ as the smallest possible improvement made by any improving 2-change in any tour, i.e.,

$$\Delta = \min_{e_1,e_2,e_3,e_4 \in E} \Delta(e_1, e_2, e_3, e_4) > 0\,.$$

Problem 1

Show that 2-Opt performs at most n/Δ improving 2-changes before finding a local optimum, i.e., no further improving 2-change is possible.

Problem 2

Show that $\mathbf{Pr}[\Delta \le \varepsilon] \le n^4 \varepsilon \phi$ for any $\varepsilon > 0$.

Problem 3

Use Problem 1, Problem 2, and an adequate worst-case bound to show that the expected number of improving steps performed by 2-Opt is in $O((n^4 \cdot n \cdot \phi) \cdot \log(n!)) = O(n^6 \log(n) \cdot \phi)$.

Hint: For a random variable X which takes only non-negative integer values, the expected value can be written as $\mathbf{E}[X] = \sum_{i=1}^{\infty} \mathbf{Pr}[X \ge i]$.