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Probabilistic Analysis
of Algorithms
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Problem Set 6

Problem 1

Consider the following input model for problems whose input consists of a sequence of numbers whose permutation determines the running time of an algorithm: First, the adversary may select a 'bad' configuration of the data (i.e. a bad ordering) S. In a second step this input is slightly modified by independently picking each index of the input sequence with probability p and then applying a random permutation on the elements picked. We call this modified sequence S'. This model is called *partial permutation*.
For example the adversary might pick the input data $S = (s_1 = 1, s_2 = 2, \ldots, s_{10} = 10)$ for quicksort. The indices selected when picking each one with a probability $p = \frac{1}{3}$ could be s_3, s_4, s_9 and the random permutation on these elements could be $(2, 3, 1)$. Applying it to the sequence would yield the modified input $S' = (1, 2, 4, 9, 5, 6, 7, 8, 3, 10)$.

Assume we have a fixed sequence $S = (s_1, \ldots, s_n)$ and the probability for picking each index is p.

a) What is the expected number of indices being picked?

b) How big is the probability to obtain some fixed modification S' of S in which k indices are permuted?

Problem 2

Consider an implementation of quicksort that recursively picks the first element a of a sequence as pivot element and splits the sequence into two lists of elements that are smaller (bigger) than a. The input consists of a permutation of the elements $\{1, \ldots, n\}$. We want to estimate the total number of comparisons C quicksort will make during sorting the input sequence. We denote by X_{ij} the variable that indicates whether i and j are compared with i being the pivot element. Obviously $C = \sum_{i,j} X_{ij}$.
Consider the following observation: $X_{ij} = 1$ iff in S' the element i is the first of all elements with values between i and j.

1. For a random ordering of the elements and any i, j from $\{1, \ldots n\}$, what is the probability of X_{ij} to be 1?

2. Give an upper bound of the expected size of C if the input is a random permutation of n elements.

Problem 3

A useful tool for bounding the sum of independent random variables is the *Chernoff Bound*: If $X_1, \ldots, X_n$ are independent random variables taking values from $\{0, 1\}$ and $X = \sum_{i=1}^{n} X_i$, then

$$\mathbf{Pr}[X < (1 - \delta)E[X]] < e^{-\frac{\delta^2}{2}E[X]}$$

for any $0 < \delta < 1$.
Assume the adversary chose an input sequence S for quicksort and i is one of the selected elements during the succeeding step of partial permutation. Try to find an upper bound for $\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{Pr}[X_{ij} = 1]$.
Hint: Let $l = |i - j|$. Consider two different cases: The first one, where the number of selected elements between i and j is at most $\frac{lp}{2}$, versus the second one, where it is bigger.

Problem 4

Now we look at the opposite case: the adversary specified S and i is *not* selected by the subsequent partial permutation.
For any $i < j$ let's look at three sets:

- $S_1 = \{x \mid x \leq i\}$, $|S_1| = i$
- $S_2 = \{x \mid i < x \leq j\}$, $|S_2| = j - i$
- $S_3 = \{x \mid j < x\}$, $|S_3| = n - j$

Assume that for each S_i at least $\frac{p|S_i|}{2}$ elements are chosen and thus will be permuted. Give an upper bound for $\mathbf{Pr}[X_{ij}]$ in this case.

Considering the results you obtained when solving the previous exercises, what bound on the expected value of C can be derived in the model of partial permutation? How does this bound compare to the bound you obtained for the average case (Problem 2) and the worst-case bound of quicksort, which is $\mathcal{O}(n^2)$?