Problem Set 6

Problem 1

Consider the following input model for problems whose input consists of a sequence of numbers whose permutation determines the running time of an algorithm: First, the adversary may select a ‘bad’ configuration of the data (i.e. a bad ordering) $S$. In a second step this input is slightly modified by independently picking each index of the input sequence with probability $p$ and then applying a random permutation on the elements picked. We call this modified sequence $S'$. This model is called partial permutation.

For example the adversary might pick the input data $S = (s_1 = 1, s_2 = 2, \ldots, s_{10} = 10)$ for quicksort. The indices selected when picking each one with a probability $p = \frac{1}{2}$ could be $s_3, s_4, s_9$ and the random permutation on these elements could be $(2, 3, 1)$. Applying it to the sequence would yield the modified input $S' = (1, 2, 4, 9, 5, 6, 7, 8, 3, 10)$.

Assume we have a fixed sequence $S = (s_1, \ldots, s_n)$ and the probability for picking each index is $p$.

a) What is the expected number of indices being picked?

b) How big is the probability to obtain some fixed modification $S'$ of $S$ in which $k$ indices are permuted?

Problem 2

Consider an implementation of quicksort that recursively picks the first element $a$ of a sequence as pivot element and splits the sequence into two lists of elements that are smaller (bigger) than $a$. The input consists of a permutation of the elements $\{1, \ldots, n\}$. We want to estimate the total number of comparisons $C$ quicksort will make during sorting the input sequence. We denote by $X_{ij}$ the variable that indicates whether $i$ and $j$ are compared with $i$ being the pivot element. Obviously $C = \sum_{i,j} X_{ij}$.

Consider the following observation: $X_{ij} = 1$ iff in $S'$ the element $i$ is the first of all elements with values between $i$ and $j$.

1. For a random ordering of the elements and any $i, j$ from $\{1, \ldots, n\}$, what is the probability of $X_{ij}$ to be 1?

2. Give an upper bound of the expected size of $C$ if the input is a random permutation of $n$ elements.
Problem 3
A useful tool for bounding the sum of independent random variables is the Chernoff Bound: If $X_1, \ldots, X_n$ are independent random variables taking values from $\{0, 1\}$ and $X = \sum_{i=1}^{n} X_i$, then
\[
\Pr[X < (1 - \delta)E[X]] < e^{-\frac{\delta^2}{2}E[X]}
\]
for any $0 < \delta < 1$. Assume the adversary chose an input sequence $S$ for quicksort and $i$ is one of the selected elements during the succeeding step of partial permutation. Try to find an upper bound for $\sum_{i=1}^{n} \sum_{j=1}^{n} \Pr[X_{ij} = 1]$.

*Hint:* Let $l = |i-j|$. Consider two different cases: The first one, where the number of selected elements between $i$ and $j$ is at most $\frac{ln}{2}$, versus the second one, where it is bigger.

Problem 4
Now we look at the opposite case: the adversary specified $S$ and $i$ is not selected by the subsequent partial permutation.

For any $i<j$ let’s look at three sets:

- $S_1 = \{x \mid x \leq i\}$, $|S_1| = i$
- $S_2 = \{x \mid i < x \leq j\}$, $|S_2| = j - i$
- $S_3 = \{x \mid j < x\}$, $|S_3| = n - j$

Assume that for each $S_i$ at least $\frac{n|S_i|}{2}$ elements are chosen and thus will be permuted. Give an upper bound for $\Pr[X_{ij}]$ in this case.

Considering the results you obtained when solving the previous exercises, what bound on the expected value of $C$ can be derived in the model of partial permutation? How does this bound compare to the bound you obtained for the average case (Problem 2) and the worst-case bound of quicksort, which is $O(n^2)$?