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Probabilistic Analysis
of Algorithms
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Problem Set 4

Problem 1

Imagine you are working on a database whose entries consist of data about the students of the University of Bonn. Your task is to implement a function that sorts all entries with respect to the year of birth. How and why can this task be solved by an algorithm that is faster than $\Theta(n \log n)$?

Problem 2

Let an instance of the knapsack problem with capacity t, weights $w_1, \dots, w_n$ and profits $p_1, \dots, p_n$ from the interval $[0, 1]$ be given. As always the density functions that determine p_i are limited by ϕ. Recall the definition of the winner gap:

$$\Delta := px^* - px^{**},$$

where

$$x^* := \arg\max\{px \mid x \in \{0,1\}^n \text{ and } wx \le t\}$$

$$x^{**} := \arg\max\{px \mid x \in \{0,1\}^n \text{ and } wx \le t \text{ and } x \neq x^*\}.$$

We assume that there are at least two valid solutions such that Δ is well-defined.
If you could be sure that $\Delta > n2^{-\ell}$ for some $\ell \in \mathbb{N}$, in what respect would this knowledge help you to find an optimal solution quickly?

Hint: Think about how rounding the profits influences the solution of the problem. Remember that it is possible to solve the integer version of the knapsack problem in $\mathcal{O}(nP)$, where P is the sum of all profits.

Problem 3

Assume you want to solve a general instance I of the knapsack problem as given above but the only tool you may use is a solver for rounded instances $\lfloor I \rfloor$, where the binary value of p_i is cut off after b bits.

1. Give a sufficient condition that can be tested efficiently for the event that the optimal solution x' of $\lfloor I \rfloor$ coincides with the optimal solution x^* of I.

2. Use this condition to derive an algorithm to solve the knapsack problem and analyse its smoothed running time.

Hint: Look at the following modification $\tilde{I}$ of I:

$$\tilde{p_i} = \begin{cases} \lfloor p_i \rfloor_b & \text{if } x'_i = 1, \\ \lceil p_i \rceil_b & \text{else,} \end{cases}$$

where $\lceil p_i \rceil_b := \lfloor p_i \rfloor_b + 2^{-b}$.

Try to figure out the relation between the solutions of $\lfloor I \rfloor$ and $\tilde{I}$. How does the size of the winner gap affect this relation? Remember that from Problem Set 3 you already know that

$$\Pr\left[\Delta \leq \varepsilon\right] \leq n\phi\varepsilon.$$

Problem 4

Suppose you want to draw a random integer from $\{1, ..., n\}$. The only device you have at hand is a regular ℓ-sided die. Think of a way to use this die as a fair random number generator. Note that n does not have to be ℓ^k for a k in $\mathbb{N}$.

a) What would be the worst case number of tosses you need to generate one random number?

b) What is the expected value of the number of trials you have to make until you get one valid number?

c) Try to find a strategy where the expected number of tosses for $n = 19$ and $\ell = 6$ is lower than 3.5.