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Probabilistic Analysis
of Algorithms
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Problem Set 3

Problem 1

Find an algorithm for the knapsack problem that runs in the worst case in time $O(nP)$, where n is the number of items, all profits $p_1, \ldots, p_n \in \mathbb{N}$ are natural numbers, and $P := \sum_{i=1}^{n} p_i$. Why does the existence of such an algorithm not prove $\mathcal{P} = \mathcal{NP}$?

Problem 2

We denote by the *winner gap* Δ the difference of quality between the best solution x^* and the second best solution x^{**} for a given problem. So for the knapsack problem with n items, weights $w_1, \ldots w_n \in \mathbb{R}_{>0}$, and the capacity t we define:

$$\Delta := px^* - px^{**},$$

where

$$x^* := \arg\max\{px \mid x \in \{0,1\}^n \text{ and } wx \leq t\}$$

$$x^{**} := \arg\max\{px \mid x \in \{0,1\}^n \text{ and } wx \leq t \text{ and } x \neq x^*\}.$$

We assume that there are at least two feasible solutions, such that Δ is well-defined. Furthermore let each profit p_i be chosen independently and at random according to the probability density $f_i : [0,1] \to [0,\phi]$ for some fixed $\phi \geq 1$. Show that for any $\epsilon > 0$

$$\mathbf{Pr}[\Delta \leq \varepsilon] \leq n\phi\varepsilon.$$

Problem 3

Let E be a set of n elements and F be a family of subsets of E (i.e. F is a subset of the power set of E). Suppose each element $x \in E$ is independently assigned a weight $w(x)$ uniformly at random from the set $\{1, \ldots, N\}$. Let the weight of a set $S \in F$ be defined as

$$w(S) := \sum_{x \in S} w(x).$$

Let S^* be one element of F with maximum weight, i.e. there is no other element in F whose weight is higher than $w(S^*)$. Prove the following statement:

$$\mathbf{Pr}[\exists S' \in F \backslash \{S^*\} : w(S') = w(S^*)] < \frac{n}{N}.$$

In other words: the probability of S^* to be a unique maximum is at least $1 - \frac{n}{N}$.

Problem 4

In a quiz show three participants can win a trip to Hawaii if they win the following game: Each participant gets independently and uniformly at random either a red or a green hat; he cannot see the color of his hat but he sees the colors of the others. Then, without communicating, all three write down either “red”, “green” or “unknown”. The three players win if at least one player wrote down “red” or “green” and if all players that wrote down “red” or “green” correctly guessed the color of their own hat.

(a) Give a strategy for the three players that guarantees a chance for winning of exactly 50%.

(b) Is there a scheme that guarantees a winning probability of more than 50%?